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Locality, Quantum Many-Body Dynamics, and Gapped Ground State Phases.

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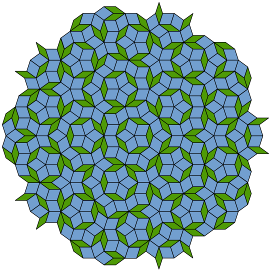
Outline

I.a. Quantum spin systems

Quantum spin systems are defined on a 'lattice', some nice discrete metric space (Γ, d), such as $\mathbb{Z}^\nu$ with the usual ℓ^1 distance.

A useful notion of 'nice' is a power law bound on the size of balls: $|B_x(R)| \leq C(1 + R^\nu)$, for all $x \in \Gamma, R \geq 0$.



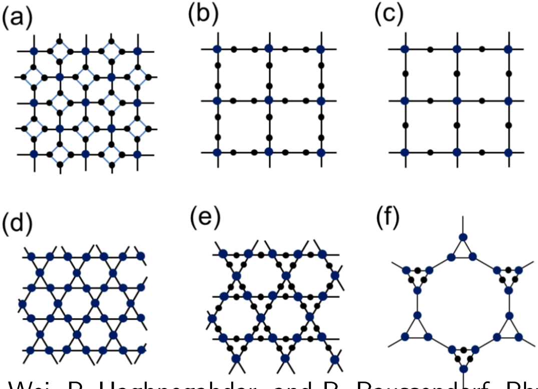
Penrose tiling

Delone:

$\Gamma \subseteq \mathbb{R}^\nu$ $\quad \exists \quad R_0, R_1 > 0$

$\forall x \in \Gamma \quad B_x(R) = \emptyset$ if $R < R_0$

$B_x(R) \neq \emptyset$ if $R > R_1$



From: T.-C. Wei, P. Haghnegahdar, and R. Raussendorf, Phys. Rev. A **90** (2014), 042333

$\mathcal{B}(\mathbb{C}^{n_x})$

$\shortparallel$

For each $x \in \Gamma$, observables are $n_x \times n_x$ matrices: $\mathcal{A}_{\{x\}} = M_{n_x}(\mathbb{C})$, $n_x \geq 2$.
For finite $\Lambda \subset \Gamma$,

$$\mathcal{A}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{A}_{\{x\}}.$$

For $\Lambda_1 \subset \Lambda_2$, $\mathcal{A}_{\Lambda_1}$ is naturally embedded into $\mathcal{A}_{\Lambda_2}$ and therefore we can define

$$\mathcal{A}_{\mathrm{loc}} = \bigcup_{\text{finite } \Lambda \subset \Gamma} \mathcal{A}_\Lambda, \quad \mathcal{A}_\Gamma = \overline{\mathcal{A}_{\mathrm{loc}}}^{\|\cdot\|}.$$

$A = A_\Lambda \otimes \mathbb{1}_{\Lambda^c}$

$A \in \mathcal{A}_\Lambda$ is said to be supported in Λ, and the support of A is the smallest Λ for which this holds.

A quantum spin model is typically defined in terms of an interaction: $\Phi(X) = \Phi(X)^* \in \mathcal{A}_X$, for all finite $X \subset \Gamma$, and local Hamiltonians

$$H_\Lambda = \sum_{X \subset \Lambda} \Phi(X),$$

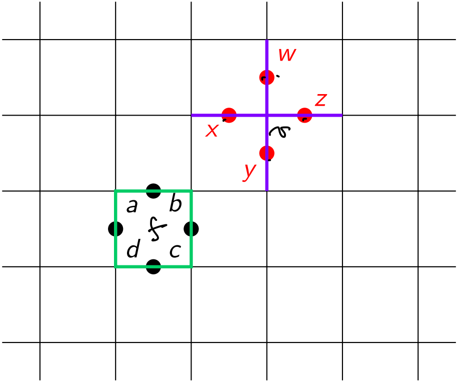
Examples: nearest neighbor spin models such as Heisenberg chain, AKLT models, have $\Gamma = \mathbb{Z}^\nu$, $\Phi(X) \neq 0$ only for $X = \{x, y\}$ with $d(x, y) = 1$.

Heisenberg model: $H = \sum_{\substack{x,y \\ |x-y|=1}} J \, \bar{S}_x \cdot \bar{S}_y \leftarrow$ n_x, n_y - dimensional spin matrices.

AKLT chain: $\Gamma = \mathbb{Z}$, $n_x = 3$ $\forall x \in \mathbb{Z}$. Spin-1 $= P^{(2)}_{x,x+1}$

PRL 1987 $\quad H = \sum_x \left(\frac{1}{2} \bar{S}_x \cdot \bar{S}_{x+1} + \frac{1}{6} (S_x \cdot S_{x+1})^2 + \frac{1}{3} \mathbb{1} \right)$

Toric Code Hamiltonian (Kitaev 2006):
$\Gamma = \mathcal{E}(\mathbb{Z}^2)$, the edges of the square lattice; $\mathcal{A}_x = \mathbb{C}^2$, for all $x \in \Gamma$



$$H = \sum_v (\mathbb{1} - A_v)$$

$$+ \sum_f (\mathbb{1} - B_f)$$

$$A_v = \sigma^1_w \sigma^1_x \sigma^1_y \sigma^1_z$$

$$B_f = \sigma^3_a \sigma^3_b \sigma^3_c \sigma^3_d$$

Heisenberg dynamics of finite systems:

$$\tau_t^\Lambda(A) = U_\Lambda(t)^* A U_\Lambda(t), \quad U_\Lambda(t) = e^{-itH_\Lambda}, A \in \mathcal{A}_\Lambda.$$

If the interaction depends on time, $t \mapsto \Phi(X,t) \in \mathcal{A}_X$, say continuously, then

$$\begin{aligned} \frac{d}{dt} U_\Lambda(t,s) &= -iH_\Lambda(t) U_\Lambda(t,s) \\ U_\Lambda(s,s) &= \mathbb{1}, \end{aligned}$$

defines cocycles of unitaries $U_\Lambda(t,s)$ and automorphisms $\tau_{t,s}^\Lambda(A) = U_\Lambda(t,s)^* A U_\Lambda(t,s)$.

Alternatively (and more generally if no uniqueness result for the IVP exists), a solution can be constructed using the Dyson series:

$$U(t,s)\psi = \psi + \sum_{n=1}^\infty (-i)^n \int_s^t \int_s^{t_1} \cdots \int_s^{t_{n-1}} H(t_1)\cdots H(t_n)\psi \, dt_n \cdots dt_1 \, .$$

$$= T\left[\exp i \int_s^t H(s) ds \right] \psi$$

Locality, Quasi-Locality, Almost-Locality

Locality is a crucial notion for many-body systems. Observables in $\mathcal{A}_{\mathrm{loc}}$ are called local, those in $\mathcal{A}_\Gamma$ quasi-local.

By construction, for all $A \in \mathcal{A}_\Gamma$ and any sequence $\Lambda_n \uparrow \Gamma$, there exist $\mathcal{A}_{\Lambda_n} \ni A_n \to A$. A concrete sequence of local approximations of any $A \in \mathcal{A}_\Gamma$ can be obtained by using the conditional expectations Π_Λ:

$$\Pi_\Lambda = \mathrm{id}_{\mathcal{A}_\Lambda} \otimes \rho \restriction_{\mathcal{A}_{\Gamma \setminus \Lambda}}, \text{ where } \rho \text{ is the tracial state.}$$

$\mathrm{Tr} A \in \mathbb{C} \cong \mathbb{C}\mathbb{1}$

$A \in \mathfrak{A}_\Lambda$

$C \equiv \int du\, u^* A u = (\tilde{\mathrm{Tr}} A)\mathbb{1}$

normalized Haar measure

$\mathcal{U}(\mathfrak{A}_\Lambda)$

all unitaries in $\mathfrak{A}_\Lambda$

$$C \equiv \int du\, u^* A u$$

$$V^* C V = \int du\, V^* u^* A u V$$

$$= \int du V (V u)^* A u V) = C$$

$\Rightarrow$ $C = c\,\mathbb{1}$, $c = \hat{\mathrm{Tr}}\, A$

by Schur's Lemma

$$\pi_\Lambda = \mathrm{id}_\Lambda \otimes \hat{\mathrm{Tr}}_{\Lambda^c}$$

$\Rightarrow$ $\forall A \in \mathcal{O}_\Gamma$, $\pi_\Lambda(A) \in \mathcal{O}_\Lambda$

Pick $\Lambda_n \subset \Lambda_{n+1}$, $\bigcup_n \Lambda_n = \Gamma$

For any f, positive and decreasing to 0, we can define

$$\|A\|_f = \|A\| + \sup_n f(n)^{-1} \|A - \Pi_{\Lambda_n}(A)\|.$$

Then, $\mathcal{A}_f = \{A \in \mathcal{A}_\Gamma | \|A\|_f < \infty\}$ is a Banach *-algebra for this norm (e.g., Moon-Ogata 2019).

Some authors prefer the notion of almost locality: $A \in \mathcal{A}_\Gamma$ is almost local, if $A \in \mathcal{A}_f$ for some f decaying faster than any power (Kapustin-Sopenko 2020). The set of all almost local A is a Fréchet *-algebra.

Lieb-Robinson bounds (Lieb-Robinson, CMP 1972) provide an estimate for commutators. Due to the following inequalities, they are useful to measure locality of observables:

$$- \pi_\Lambda(A)$$

$$\|A - \Pi_\Lambda(A)\| \leq \sup_{B \in \mathcal{A}_{\Gamma\setminus\Lambda}, \|B\|=1} \|[A, B]\| \leq 2\|A - \Pi_\Lambda(A)\|.$$

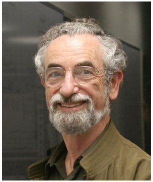
$$A - \pi_\Lambda(A) = A \int_{\mathcal{U}(\mathcal{O}_{\Lambda^c})} dU \, U^* U - \int_{\mathcal{U}(\mathcal{O}_{\Lambda^c})} dU \, U^* A U$$

$$= \int dU \, U^* [UA - AU]$$

$$\| \quad \| \leq \int dU \, \|U^+ [A, U]\| = \int dU \, \|[A, U]\|$$

$$\leq \sup_{B, \|B\|=1} \|[A, B]\|$$

Commun. math. Phys. 28, 251—257 (1972)



The Finite Group Velocity of Quantum Spin Systems

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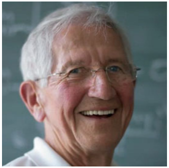

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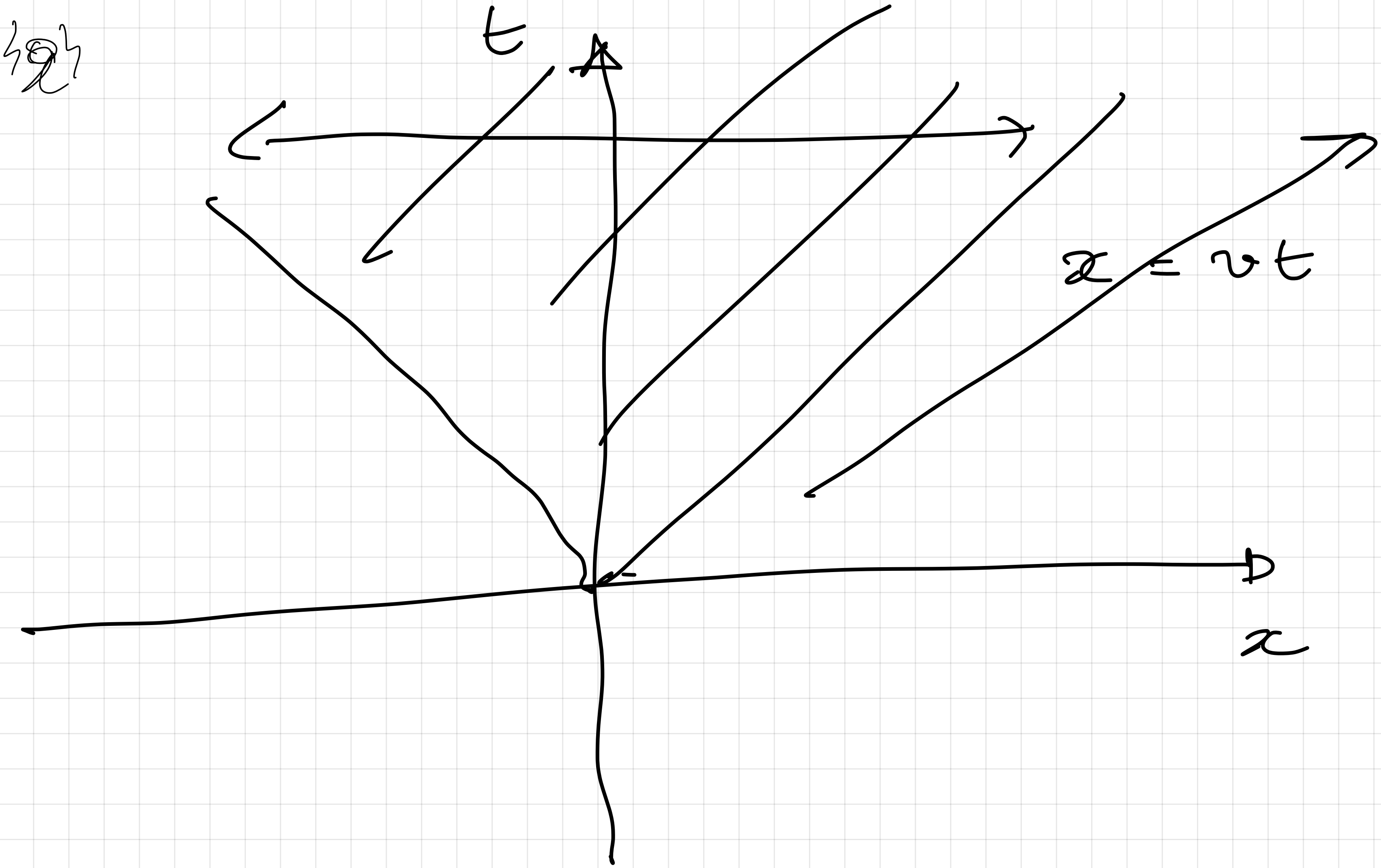
Abstract. It is shown that if Φ is a finite range interaction of a quantum spin system, τ_t^{Φ} the associated group of time translations, τ_x the group of space translations, and A, B local observables, then

$$\lim_{\substack{|t| \to \infty \\ |x| > v|t|}} \| [\tau_t^{\Phi} \tau_x(\mathrm{A}), \mathrm{B}] \| \, e^{\mu(v) t} = 0$$

whenever v is sufficiently large ($v > V_{\Phi}$) where $\mu(v) > 0$. The physical content of the statement is that information can propagate in the system only with a finite group velocity.

$A \in \mathcal{O}_{\{0\}}$ $\quad \|[\tau_t(A), B]\| \le C\|A\|\|B\| e^{+a(v|t| - d(x,y))}$

$B \in \mathcal{O}_{\{x\}}$



Telescopic sums: given $\Lambda_n \uparrow \Gamma$ and decay function f, $A \in \mathcal{A}_f$, consider the identity

$$A = \Pi_0(A) + \left[\sum_{n=1}^{N} \Pi_n(A) - \Pi_{n-1}(A)\right] + A - \Pi_N(A)$$

$-A + A$

Note $\|\Pi_n(A) - \Pi_{n-1}(A)\| \leq \|A\|_f(f(n) + f(n-1))$. If f is summable, we obtain an absolutely convergent series for A:

$$A = \Pi_0(A) + \sum_{n=1}^{\infty}[\Pi_n(A) - \Pi_{n-1}(A)].$$

II. Lieb-Robinson bounds

Let $F : [0,\infty) \to (0,\infty)$ be decreasing with the properties, for all $x, y \in \Gamma$

$$\sum_{z \in \Gamma} F(d(x,z))F(d(z,y)) \leq (C_F) F(d(x,y))$$

$$\|F\| := \sup_{x \in \Gamma} \sum_{y \in \Gamma} F(d(x,y)) < \infty$$

For such F, which are called F-functions, define a norm on interactions:

$$\|\Phi\|_F = \sup_{x,y \in \Gamma} F(d(x,y))^{-1} \sum_{X, x,y \in X} \|\Phi(X)\|.$$

Examples: If $\Gamma = \mathbb{Z}^\nu$, $F(r) = (1+r)^{-\nu'}$, for any $\nu' > \nu$ is an F-function.
In general, if F is an F-function, then so are

$$F(r) e^{-a g(r)}, \text{ with } a \geq 0, \text{ and } g(r) = r^\theta, \theta \in [0,1], \text{ or } g(r) = r/(\log(1+r))^2.$$

One defines corresponding Banach space of possibly time dependent interactions: $\mathcal{B}_F$, with uniform norm $|||\Phi|||_F$;
$\mathcal{B}_{a,\theta}, \mathcal{B}_{a,\theta}(I), \mathcal{B}^1_{a,\theta}(I), I \subset \mathbb{R}, \ldots$

Some useful estimates in terms of F-norms.

$$\begin{aligned}
\sum_{\substack{X \subset_f \Gamma: \\ x,y \in X}} \|\Phi(X)\| &\leq \|\Phi\|_F F(d(x,y)) \\
\sum_{\substack{X \subset_f \Gamma: \\ x \in X}} \|\Phi(X)\| &\leq \|F\| \|\Phi\|_F \\
\sum_{\substack{X \subset_f \Gamma: \\ X \cap Y \neq \emptyset}} \|\Phi(X)\| &\leq |Y| \|F\| \|\Phi\|_F \\
\sum_{\substack{X \subset_f \Gamma: \\ X \cap Y \neq \emptyset, X \cap Z \neq \emptyset}} \|\Phi(X)\| &\leq |Y||Z| \|\Phi\|_F F(d(Y,Z)) \\
\sum_{\substack{X \subset_f \Gamma: \\ X \cap Y \neq \emptyset, X \cap Z \neq \emptyset}} \|\Phi(X)\| &\leq \|\Phi\|_F \sum_{y \in Y, z \in Z} F_0(d(y,z)) e^{-ag(d(Y,Z))} \\
&\leq \|\Phi\|_F \min(|Y|, |Z|) \|F_0\| e^{-ag(d(Y,Z))}
\end{aligned}$$

For $X \subset \Gamma$, $\partial_\Phi X$ is the set spins in X that interact with the complement of X:

$$\partial_\Phi X = \#\{x \in X | \exists Y \subset \Gamma, \Phi(Y) \neq 0, X \cap Y \neq \emptyset, Y \cap (\Gamma \setminus X) \neq \emptyset\}.$$

Theorem (N-Sims-Young, JMP 2019) $\mathcal{Z} = \mathbb{R}, \quad I = [a, b]$

Let $\Phi \in \mathcal{B}_F(I)$ and X, Y finite subsets of Γ with $X \cap Y = \emptyset$, $A \in \mathcal{A}_X, B \in \mathcal{A}_Y$, and $s, t \in I$, we have

$$\left\| [\tau^\Phi_{t,s}(A), B] \right\| \leq 2\|A\|\|B\| \left(e^{2 \int_{\min(s,t)}^{\max(s,t)} \|\Phi\|_F(r) dr} - 1 \right) D(X, Y)$$

and the quantity $D(X, Y)$ is given by

$$D(X, Y) = \min \left\{ \sum_{x \in X} \sum_{y \in \partial_\Phi Y} F(d(x, y)), \sum_{x \in \partial_\Phi X} \sum_{y \in Y} F(d(x, y)) \right\}.$$

If $F = F_0 e^{-g}$, D can be estimate by

$$\begin{aligned} D(X,Y) &\leq \min\{|\partial_\Phi X|, |\partial_\Phi Y|\}\|F_0\| e^{-g(d(X,Y))} \\ &\leq \min(|X|,|Y|)\|F_0\| e^{-g(d(X,Y))} \end{aligned}$$

For time-independent $\Phi \in \mathcal{B}_F$ and $F(r) = e^{-ar} F_0$ one easily recovers the familiar form of the Lieb-Robinson bounds as follows:

$$\left\|[\tau_t^\Phi(A), B]\right\| \leq C\|A\|\|B\| \min(|X|,|Y|) e^{a(v_{LR}|t| - d(X,Y))},$$

with

$$v_{LR} = 2a^{-1}\|\Phi\|_F, \quad C = 2\|F_0\|_1.$$

The LRBs is stated without reference to the finite volume Λ. The convergence of the limit

$$\tau^{\Phi}_{t,s}(A) = \lim_{\Lambda \to \Gamma} \tau^{\Phi,\Lambda}_{t,s}(A)$$

can itself be proved using LRBs for finite-volume dynamics $\tau^{\Phi,\Lambda}_{t,s}$, and the infinite-volume dynamics inherits the bounds.

Unless F decays as an exponential this is the only known way to prove existence of the thermodynamic limit of the dynamics for general interactions.

Theorem

Let $\Phi \in \mathcal{B}_F(\Gamma)$. Along any increasing, exhaustive sequence $\{\Lambda_n\}$ of finite subsets of Γ, the norm limit

$$\tau_t^{\Phi}(A) = \lim_{n\to\infty} \tau_t^{\Phi,\Lambda_n}(A)$$

exists for all $t \in \mathbb{R}$ and $A \in \mathcal{A}_\Gamma^{\mathrm{loc}}$. The convergence is uniform for t in compact sets, and the limit it is independent of the choice of exhaustive sequence $\{\Lambda_n\}$. The set $\{\tau_t^{\Phi}\}_{t\in\mathbb{R}}$ defines a strongly continuous one-parameter group of $*$-automorphisms of $\mathcal{A}_\Gamma$.

The same result holds for $\Phi \in \mathcal{B}_F([a, b])$, but with the group property replaced by a composition property.

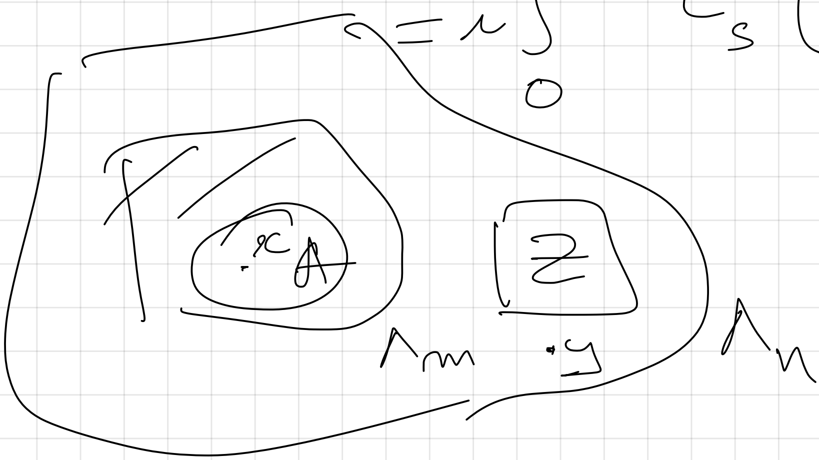
proof. (up to exercise) $A \in \mathcal{O}_{loc}$, $\operatorname{supp} A \subset \Lambda_m$

$\left(\tau_t^{\Lambda_n}(A)\right)_n$ is Cauchy; $\forall n > m$

$$\tau_t^{\Lambda_n}(A) - \tau_t^{\Lambda_m}(A) = \tau_s^{\Lambda_n} \circ \tau_{t-s}^{\Lambda_m}(A) \Big|_0^t$$

$$= \int_0^t \frac{d}{ds}\left(\tau_s^{\Lambda_n} \circ \tau_{t-s}^{\Lambda_m}(A) \right) ds$$

$$= i \int_0^t \tau_s^{\Lambda_n}\left(\left[H_{\Lambda_n} - H_{\Lambda_m}, \tau_{t-s}^{\Lambda_m}(A) \right] \right) ds$$



$$= i \sum_{\substack{Z \subset \Lambda_n \\ Z \cap \Lambda_m \neq \emptyset \\ Z \cap (\Lambda_n \setminus \Lambda_m) \neq \emptyset}} \int_0^t \left[\Phi(Z), \tau_{t-s}^{\Lambda_m}(A) \right] ds$$

$$H_\Lambda = \sum_{X \subset \Lambda} \Phi(X)$$

$\boxed{Ex}$ $\| \tau_t^{\Lambda_n}(A) - \tau_t^{\Lambda_m}(A) \|$ $\quad \underline{A \in \mathfrak{A}_X}$

$$\leq 2 \|A\| \, \|\Phi\|_F \, e^{2|t| \|\Phi\|_F}$$

$$* \sum_{x \in X} \underline{\sum_{y \in \Lambda_n \setminus \Lambda_m}} F(d(x,y))$$

$$\leq \cdots |X| \sum_{y \in \Gamma \setminus \Lambda_m} F(d(x,y)) \downarrow 0$$

on $\Lambda_m \nearrow \Gamma$

$$\|F\| = \sup_x \sum_y F(d(x,y)) < +\infty$$

Applications of LRB and quasi-locality

The use of LRBs to derive and apply quasi-locality properties of quantum many-body dynamics and related transformations is ubiquitous.

1. Exponential decay of correlations in gaped ground states (N-Sims & Hastings-Koma, CMP 2006).
2. Stability of the ground state gap. 'Local Perturbations Perturb Locally'.
Bravyi-Hastings-Michalakis 2010, 2011, 2013, Bachmann-Michalakis-N-Sims 2012, N-Sims-Young 2019-2020
3. Quantized Hall currents in interacting systems, many-body adiabatic theorems
Hastings-Michalakis 2015, Bachmann-Bols-DeRoeck-Fraas 2017-2021, Monaco-Teufel 2019, Henheik-Teufel 2020
4. Robustness of anyon character of excitations in quantum-double models.
Cha-Naaijkens-N, 2018, 2020
5. Classifying Symmetry Protected Topological (SPT) Phases.
Matsui 2010, Bachmann-Michalakis-N-Sims 2012, Ogata 2018-21, Sopenko 2021, Kapustin-Sopenko-Yang 2021

The quasi-adiabatic evolution

A key object for many applications is the The quasi-adiabatic evolution aka spectral flow.

Suppose Φ_0 and Φ_1 are two interactions with an interpolating differentiable curve $\Phi(s)$, $s \in [0,1]$, in a Banach space with sufficient decay (for concreteness, say, $F(r) = F_0(r)e^{-ar^\theta}$). Then, there is an equivalent curve of interactions, also denoted by Φ, that is supported on balls s.t.

$$\Phi_x(s) := \sum_{n \geq 0} \Phi(s, b_x(n)) \in \mathcal{A}_f,$$

for a suitable f of stretched exponential decay.

The Hastings generator of the 'quasi-adiabatic evolution' (Hastings 2004, Hastings-Wen 2005, Bachmann-Michalakis-N-Sims 2012) is defined by the 'interaction'

$$\tilde{\Psi}_x(s) = \int_{-\infty}^{\infty} w_a(t) \int_0^t \tau_u^{\Phi(s)} \left(\frac{d}{ds} \Phi_x(s) \right) du\, dt$$

with $w_a(t)$ a specific function of almost exponential decay $\sim e^{-\frac{a|t|}{(\log a|t|)^2}}$, with $a > 0$.

Using LRBs, we can show $\tilde{\Psi}_x(s) \in \mathcal{A}_f$, for a stretched exponential f.

Using a telescopic sum and conditional expectations $\Pi_{b_x(n)}$, we can construct a true interaction $\Psi \in \mathcal{B}_{a',\theta}$, equivalent to $\tilde{\Psi}$.

$\alpha_s = \tau_{s,0}^{\Psi} \qquad s \in [0,1]$

Theorem (Bachmann, Michalakis, N, Sims, 2012)

(i) The automorphisms α_s for $s \in [0,1]$, generated by $\Psi(s)$ with s as the 'time'-parameter, are a strongly continuous cocycle of quasi-local automorphisms, satisfying Lieb-Robinson bounds with F of stretched exponential decay.
(ii) If, in addition, Φ_0 and Φ_1 and the interpolating differentiable curve $\Phi(s)$ are interactions with a unique gapped ground state ω_s with gap $\geq \gamma > 0$, and we pick $a < 2\gamma/7$ in w_a, we have $\omega_s = \omega_0 \circ \alpha_s, \quad s \in [0,1]$.

" $\langle \psi_0(s), A \psi_0(s) \rangle = \langle \psi_0(0), \alpha_s(A) \psi_0(0) \rangle$

- Lieb-Robinson bounds are essential to construct true interaction and to show existence of the thermodynamic limit.
- α_s inherits any symmetries of the curve $\Phi(s)$.
- Uniqueness of the ground state can be relaxed (see later).
- Decay classes other than stretched exponentials have been considered.